Finite-range scaling study of the 1D long-range Ising model

1989 J. Phys. A: Math. Gen. 22 4439

(http://iopscience.iop.org/0305-4470/22/20/020)

J. Phys. A: Math. Gen. 22 (1989) 4439–4452. Printed in the UK

Finite-range scaling study of the 1D long-range Ising model

Z Glumac and K Uzelac

Institute of Physics, University of Zagreb, Bijenička 46, POB 304, 41000 Zagreb, Croatia, Yugoslavia

Received 13 March 1989, in final form 25 May 1989

Abstract. The critical behaviour of the one-dimensional Ising model with long-range ferromagnetic interactions decaying with distance r as $1/r^{1+\sigma}$ has been studied by scaling the range of interactions. Exact calculations have been done for a system with finite ranges up to the tenth neighbour for $0<\sigma\leqslant 1$. Applying the range scaling, the critical temperature, critical exponent ν and the anomalous dimension of the order parameter have been calculated. Additional analysis of the convergence of the method has been performed by applying the Vanden Broeck and Schwartz extrapolation procedure in addition to the simple least-squares approximation and by evaluating the convergence exponents.

1. Introduction

The study of the critical behaviour of systems with long-range (LR) interactions generally requires more complex approaches than for those with short-range (SR) ones and is consequently less explored. Even such a simple model as the one-dimensional Ising model with the interactions decaying with distance r as $1/r^{1+\sigma}$ has not been solved for arbitrary σ. Non-locality of interactions also limits approaches using the renormalisation group (RG) method. The RG in reciprocal space was efficient in giving results in the form of ε-expansions around lower (Fisher *et al* 1972) or higher (Brézin *et al* 1976, Kosterlitz 1976, Bulgadaev 1984) critical dimensions. However, a number of direct space RG techniques developed for the study of the intermediate region in the case of SR interactions become inefficient for LR interactions.

In a recent letter (Uzelac and Glumac 1988) we have formulated a renormalisation approach in direct space which takes the range of the interactions as a basic scaling parameter. By analogy with finite-size scaling (FSS) (Fisher and Barber 1972, Nightingale 1976, for a review see Barber 1983) exact results for the system considered but with different finite ranges have been used to establish the scaling relations involving the exponents for the infinite-range problem. Preliminary studies using this method have been made on the Ising model with long-range interactions, where the exact solution of the infinite system with interactions truncated to a finite number of neighbours is obtained by the transfer matrix.

In the present work we continue and extend those calculations in several respects. One is to include the study of an order parameter and corresponding anomalous dimension d_ϕ, which is related to the critic exponent η and can be checked using the analytic expression $\eta=2-\sigma$, conjectured to be valid in whole long-range region of the σ. Further on, we perform more exhaustive numerical calculations providing exact results up to the ranges involving the first ten neighbours. This creates the possibility of trying out different convergence techniques and also of carying out the analysis of the convergence of results such as exists in the context of FSS (Privman and Fisher 1983).

The one-dimensional Ising model that we shall be considering is defined as

$$H=-\sum_{i,j} J_{i,j} s_i s_j \tag{1}$$

where $s_i=\pm 1$ is a classical Ising spin at site i, and $J_{i,j}=J_0/|i-j|^{1+\sigma}>0$. When the parameter σ is varied this model passes through different critical regimes. For $0<\sigma\leqslant 1$ it is known rigorously that $T_c\neq 0$ (Dyson 1969, Simon and Sokal 1981, Fröhlich and Spencer 1982) and the critical behaviour is of the long-range type. The region $\sigma<0.5$ corresponds to the mean-field (MF) region, while the region with $0.5<\sigma<1$ has a non-trivial critical behaviour which is not known exactly. Approximate results in the latter region were obtained by finite chain extrapolations (Nagle and Bonner 1970) or by ε-expansions around $\sigma=0.5$ (Fisher *et al* 1972) and $\sigma=1$ (Kosterlitz 1976). For $\sigma=1$ the transition is governed by topological defects and the critical behaviour is of the essential singularity type. The region with $\sigma>1$ corresponds to $T_c=0$ with short-range critical behaviour (Sak 1973).

The outline of the paper is the following. In the next section we explain the method of calculation and point out some differences between the present method and FSS. Section 3 is the main part of this article, where the results are presented together with the convergence analysis. The first two sub-sections deal with the extrapolation methods used and the convergence analysis. The last three sub-sections are dedicated to study of the critical temperature, the critical exponent ν and the order parameter anomalous dimension d_Φ respectively. Concluding remarks are given in § 4.

2. Method and calculations

2.1. Finite-range scaling

The idea of finite-range scaling (FRS) can be explained in the following way. Let us consider some physical quantity $C(t)$ of a system with long-range interactions J_r, exhibiting a singularity at the critical temperature T_c

$$C(t)=C_0 t^{-\rho} \tag{2}$$

where $t=(T-T_c)/T_c$, C_0 is a constant and ρ is the related critical exponent. Consider then the case where these interactions are truncated to the Nth neighbour, i.e. $J_r=0$ for $r>N$. This truncation will prevent the divergence (2) from occurring, but one can assume that for large N the new, modified, behaviour can be described by introducing a correction factor to the infinite-range behaviour. Further, invoking the scaling hypothesis, we assume this correction to be a homogeneous function of range measured by the correlation length ξ_∞ of the true long-range system, i.e.

$$C_N(t)=C_\infty(t) f(N/\xi_\infty) \tag{3}$$

where subscripts N and ∞ indicate the range of interactions. It can also be written in an equivalent form

$$C_N(t)=N^{\omega} Y(N^{1/\nu} t) \tag{4}$$

where $\omega=\rho/\nu$ is the anomalous dimension of C. In order to obtain good asymptotic behaviour of $C_N(t)$ in the limits $t=$constant, $N\to\infty$ and $t\to 0$, $N=$constant, the function $Y(x)$ has to satisfy additional constraints. When there is no phase transition in the short-range case, they can be written as

$$\lim_{x\to\infty} Y(x) = C_0 x^{-\rho} \qquad \lim_{x\to 0} Y(x) = Y_0 \tag{5}$$

where Y_0 is a constant.

The assumption (3) formulated through equations (4) and (5) is analogous to the ansatz of finite-size scaling (Fisher and Barber 1972), in which case N denotes the finite length in one or more dimensions of the system.

One may then proceed to exploit equation (3) in complete analogy with FSS. Applying equation (3) first to the correlation length leads to

$$\xi_N(t) = (N/M)\xi_M(t') \tag{6}$$

where N and M are two different ranges. The variable t' satisfies $\xi_\infty(t)/\xi_\infty(t') = N/M$ and is equal to

$$t' = (N/M)^{1/\nu} t. \tag{7}$$

The fixed point is then determined by

$$\xi_N(t^*) = (N/M)\xi_M(t^*). \tag{8}$$

Linearisation and expansion around t^* using (7) gives

$$\nu^{-1} = \ln[\xi'_N(t^*)/\xi'_M(t^*)]/\ln(N/M) - 1. \tag{9}$$

In present calculations we chose $M = N-1$, since, as in FSS, the convergence of results is better when N and M are closer to each other.

Equation (3) can be similarly applied to the order parameter Φ. At the fixed point one obtains

$$\Phi_N(t^*)/\Phi_M(t^*) = (N/M)^{-d_\phi} \tag{10}$$

where d_ϕ is the anomalous dimension of the order parameter in the true long-range system. As it describes the scaling of the order parameter at T_c, d_ϕ is directly related to the critical exponent η of the correlation function at T_c by the relation

$$d_\phi = (d-2+\eta)/2. \tag{11}$$

In order to complete the analogy between FRS and FSS we should finally mention that, for the latter, the assumption of type (3) was later justified by ε-expansion calculations (Brézin 1982). In the present case such calculations become rather complex and we do not attempt them here.

2.2. *Transfer matrix*

The applicability of the above procedure depends on the possibility of determining the exact results used in equations (6) and (10).

For the Ising chain (1) with interactions truncated at Nth neighbours as

$$J_r = \begin{cases} J_0/r^{1+\sigma} & \text{for } r \leq N \\ 0 & \text{for } r > N \end{cases} \tag{12}$$

exact calculations can be made by transfer matrix. As displayed in figure 1, for a certain range N the chain can be divided into groups of N spins so that the transfer matrix connects neighbouring groups. It is practical to choose new N-component variables α_j such that $\alpha_j(i) = S_{N(j-1)+i}$. The transfer matrix for the Hamiltonian (1) then can be written in the form

$$\mathfrak{C}_{j,j+1} = \exp\{-[H_{j,j+1} + (H_j + H_{j+1})/2]/T\} \tag{13}$$

where T is temperature (we choose the system of units where $J_0/k_B = 1$) and

$$H_{j,j+1} = -\sum_{n=0}^{N-1} J_{N-n} \sum_{i=1}^{N-n} \alpha_j(i)\alpha_{j+1}(i+n) \tag{14}$$

$$H_j = -\sum_{n=1}^{N-1} J_n \sum_{i=1}^{N-n} \alpha_j(i)\alpha_j(i+n). \tag{15}$$

The matrix $\mathfrak{C}$ is not symmetric, but has instead a symmetry property which can be expressed by

$$\mathfrak{C}_{\alpha\beta} = \mathfrak{C}_{\bar{\beta}\bar{\alpha}} \tag{16}$$

where α and β denote configurations of neighbouring columns of N spins, while $\bar{\alpha}$ and $\bar{\beta}$ are the configurations obtained by counting spins in inverse order, i.e. $\bar{\alpha}(i) = \alpha(N+1-i)$. In addition to this symmetry, $\mathfrak{C}$ is invariant to the spin reversal which reduces the original configuration space to two invariant subspaces.

Eigenvalues λ_i and eigenvectors ψ_i of $\mathfrak{C}$ have to be determined numerically. The largest matrix calculated was of order 512, which corresponds to the configurational subspace of $2^N/2$ configurations for the range $N = 10$.

Applying the standard derivation, one obtains the correlation length given by

$$\xi_N = N/\ln(\lambda_1/\lambda_2) \tag{17}$$

where λ_1 and λ_2 are the largest and second-largest eigenvalues of $\mathfrak{C}$ respectively. The additional factor N results from the fact that the application of transfer matrix l times connects two spins at a distance lN.

When studying the order parameter, a similar problem arises as in FSS. Namely, the equation of type (10) cannot be applied directly to the magnetisation since $\langle\psi_1|s_i|\psi_1\rangle = 0$ for every i whenever N is finite and becomes different from zero only by

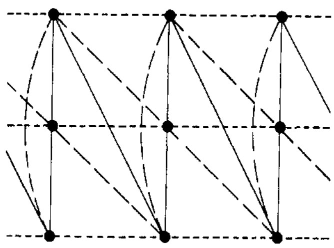

Figure 1. Chain for the case $N = 3$, drawn in zig-zag. Interactions J_1, J_2 and J_3 are represented by full, long-dashed and short-dashed lines respectively. α_j is a three-component variable describing all configurations of three spins in column j.

asymptotic degeneracy of the largest-eigenvalue state. Thus, as in FSS (Uzelac and Jullien 1981), we shall consider the quantity $\langle\psi_2|s_i|\psi_1\rangle$, which becomes equal to the magnetisation in the asymptotic limit $N\to\infty$. In addition, in order to diminish the other effects of truncation, it is convenient to consider an average over different positions of spin in a column α, i.e. we define

$$\Phi_N = \left\langle \psi_2 \left| \sum_{i=1}^{N} s_i \right| \psi_1 \right\rangle \Big/ N \tag{18}$$

to be inserted in equation (10). Notice that left and right eigenvectors in equation (18) are not equal because the transfer matrix is not symmetric. However, due to symmetry property (16), transformation from right to left eigenvectors is easily made. If $|\psi_j\rangle = \Sigma_{\{\alpha\}} A_\alpha|\alpha\rangle$, then $\langle\psi_j| = \Sigma_{\{\bar\alpha\}} A_{\bar\alpha}\langle\alpha|$, where $A_{\bar\alpha}$ are the components corresponding to configurations $\bar\alpha$.

2.3. *Mean-field region*

When establishing the analogies between FRS and FSS, one important difference is to be noticed concerning the MF region. It has been shown (Brézin 1982) that FSS breaks down for dimensions where the MF regime takes place. Without resorting to the ε-expansion formalism, one can expect that the relevance of finite-size effects will be altered there, since in the MF region the critical behaviour does not depend on dimensionality. Indeed, the calculations on the $n=\infty$ vector model (Brézin 1982) confirm that finite-size corrections to the FSS relation, analogous to equation (3), adopt a more complex form than a function of the simple ratio N/ξ_∞.

In the case of long-range interactions and FRS, it is likely that the same problem will not occur, since even in the MF region critical behaviour depends strongly on the range of the interactions. Brézin's calculations are difficult to reproduce in the present case, but with the above reasoning in mind one can make a numerical check to see if $\xi_N(T_c)$ depends linearly on N as required by the scaling ansatz (3), or otherwise. In table 1 are presented extrapolations for $\bar\xi_N = \xi_N(T_c)/N$ of model (1) in the limit $N\to\infty$ by fitting to the form $\bar\xi_N(T_c) = B + AN^{-x}$. Comparing $\bar\xi_N$ and B for different σ, one can observe the linear behaviour with no difference between the non-trivial and the MF region. This reveals a definitely different behaviour to the FSS case, suggesting

Table 1. $\bar\xi = \lim_{N\to\infty}\bar\xi_N(T_{ce})$ calculated by VBS approximation. B and x are obtained by LSA fit to the form $\bar\xi_N(T_{ce}) = B + AN^{-x}$.

σ	$\bar\xi$	B	x
0.1	0.2431	0.2415	0.98
0.2	0.2965	0.2961	0.94
0.3	0.3497	0.3493	0.91
0.4	0.4028	0.4034	0.90
0.5	0.4620	0.4621	0.84
0.6	0.5289	0.5319	0.78
0.7	0.6290	0.6320	0.75
0.8	0.7889	0.7922	0.76
0.9	1.105	1.103	1.4
1.0	—	—	—

that equation (3) should still be valid and that the range should still be a good scaling variable.

One should mention here, however, that a problem of the same type arises in the FRS on the other edge of the non-trivial region ($\sigma > 1$), where the range becomes an irrelevant variable (to be discussed in § 3).

3. Results

Before proceeding to the presentation and analysis of results, let us discuss shortly the extrapolation procedures used and the convergence analysis.

3.1. Extrapolation procedures

In an earlier paper dealing with data up to $N = 8$, we have used one simple extrapolation procedure, fitting the curve in the least-squares approximation (LSA). Fitting was made to the form

$$\rho_N = \rho + A/N^{x_\rho} \tag{19}$$

where ρ denotes the extrapolated quantity. In the MF region, the expression $(1/N)^x$ was replaced by $[(N-1)/N]^x$. In the present paper the same type of extrapolation was performed using data for $N = 8, 9, 10$.

In addition, we apply here another procedure due to Vanden Broeck and Schwartz (1979) (VBS). In order to speed up the convergence, Hamer and Barber (1981) have used it in FSS studies, making some adjustments which we follow here.

This procedure is a generalisation of the Padé approximant method. Successive approximations are given by the following recurrence relations:

$$\begin{aligned}\{[N, L+1]-[N, L]\}^{-1}+\alpha_L\{[N, L-1]-[N, L]\}^{-1} \\ =\{[N+1, L]-[N, L]\}^{-1}+\{[N-1, L]-[N, L]\}^{-1}\end{aligned} \tag{20}$$

where $[N, L]$ is the Lth-order extrapolation of ρ_N. In particular $[N, 0] = \rho_N$ and $[N, -1] = \infty$. The parameter α_L is free. For $\alpha_L = 1$ the standard Padé approximants are reproduced. For the converging data of type

$$\rho_N^{-1} = \rho^{-1} + a_1 N^{-\lambda_1} + a_2 N^{-\lambda_2} + \ldots \tag{21}$$

which is expected in our problem, it has been argued (Hamer and Barber 1981) that the best choice is $\alpha_L = ((-1)^L - 1)/2$, giving a second-order correction of $O(N^{-\lambda'})$ with $\lambda' = \min(\lambda_1 + 2, \lambda_2)$.

Therefore, the obvious advantage of the VBS method compared with the previously defined method (LSA) is its appropriateness for extrapolating from a non-monotonic sequence of values where several corrections of different order are in competition. The efficiency of this procedure is, however, limited by the fact that it requires rather high numerical precision (six to eight digits) and a sufficiently large number of data. In the present study, one can expect it to be useful for determining T_c and d_Φ, but less accurate or inapplicable for ν, where the data have been determined with less precision.

3.2. Convergence

The convergence of T_c and critical exponents as functions of N was extensively explored in the context of FSS. Within FSS those studies were also completed later by application of conformal invariance theory (since most of the problems studied were in two dimensions). In the present case, this aspect of the problem will not be considered, but the analysis of convergence, important for the reliability of the results, still remains of interest.

From the scaling hypothesis it follows that in the case of power-law critical behaviour, the convergence will generally take the power-law form, the corresponding exponent x_p (cf equation (19)) being related to the leading irrelevant critical exponent y_3. If we write down the scaling ansatz (4) in the extended form which includes the leading irrelevant field u

$$C_N(t, u) = N^{\omega} Y(N^{1/\nu} t, N^{y_3} u) \tag{22}$$

and apply it to the correlation length, then simple algebra gives the following predictions for T_c and ν (Privman and Fisher 1983):

$$t_N = (T_{cN} - T_c)/T_c = \text{constant } N^{y_3 - 1/\nu} \tag{23}$$

$$\nu_N^{-1} = \nu^{-1} + aN^{-1/\nu} t_N + bN^{y_3} = \nu^{-1} + cN^{y_3} \tag{24}$$

where a, b, c are constants in N.

Similarly applying (22) to the order parameter, one obtains for d_ϕ

$$d_{\phi N} = d_\phi + \text{constant } N^{y_3}. \tag{25}$$

The studies within FSS mentioned have most often found discrepancies between the analytic expressions predicted above and the convergence exponents calculated from finite-size data (Privman and Fisher 1983). This indicates the importance of other effects (boundary effects, other irrelevant fields) not included in equation (3).

3.3. Critical temperature

Data for T_{cN} obtained from equation (8) for $N = 4$–10 are presented in tables 2(a), 2(b) and 2(c) followed by VBS extrapolations T_{ce} made for T_{cN}^{-1}. The numerical precision of data is 10^{-10}, so it is expected that VBS extrapolations could be used with sufficient accuracy. Indeed, successive approximations show good convergence as illustrated in table 3(a) for $\sigma = 0.8$. The numerical error, which is of the same order for all the σ, is estimated to be less than 1%. T_{ce} is compared to the Nagle and Bonner (1970) results after being normalised to the ground-state energy, equal to $J_0\zeta(1+\sigma)$ ($\zeta(x)$ is the Riemann zeta function). As shown in the last two rows of tables 2(a) and 2(b), the agreement is in the first three digits. The convergence exponent x_T defined by equation (19) has been calculated by inserting into equation (19) the VBS value T_{ce} for T_c. For $\sigma < 0.9$, x_T is close to unity, showing a small minimum around $\sigma = 0.7$. For $\sigma \geqslant 0.9$ it increases abruptly. This is in strong opposition to prediction (23), since by $(1-\sigma)$ expansion (Kosterlitz 1976) we have

$$1/\nu = -y_3 = [2(1-\sigma)]^{1/2} \tag{26}$$

and both y_3 and $1/\nu$ tend to zero when $\sigma \to 1$.

This feature can be related to the change of regime occurring at $\sigma = 1$, beyond which point the range of interaction becomes irrelevant, and the short-range interaction type behaviour with $T_c = 0$ takes place. This is indicated by a change of the data T_{cN} from an ascending sequence to a descending one. The non-monotonic behaviour of T_{cN} obviously makes the expression (19) inadequate for $\sigma > 0.98$, but a calculation of the standard deviation from the form (19) (it increases by two orders of magnitude in this region) indicates that additional corrections are already important for $\sigma > 0.9$.

Table 2. Data for T_{cN} as a function of σ and N, followed by the VBS extrapolations (T_{ce}), convergence exponent (x_T) and normalised quantity $\bar{T}_{ce} = T_{ce}/\zeta(1+\sigma)$ for comparison with Nagle and Bonner (1970) results (T_c^{NB}).

N	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.5$
4	6.997 367 411	5.611 779 533	4.629 356 506	3.891 164 872	3.316 754 469
5	8.039 577 710	6.277 235 453	5.067 902 319	4.189 990 451	3.524 000 369
6	8.912 257 204	6.788 288 808	5.388 764 323	4.401 040 408	3.666 537 773
7	9.650 044 833	7.195 123 841	5.633 847 261	4.557 682 674	3.770 199 753
8	10.286 776 49	7.527 692 570	5.872 116 991	4.678 277 802	3.848 723 670
9	10.844 900 01	7.805 171 709	5.983 345 040	4.773 783 680	3.910 091 530
10	11.340 100 31	8.040 499 729	6.112 158 669	4.851 146 924	3.959 254 071
T_{ce}	19.81	10.84	7.341	5.511	4.354 7
x_T	0.96	1.05	1.15	1.20	1.23
$\bar{T}_{ce}$	—	1.938	1.867	1.774	1.667
T_c^{NB}	—	1.931	1.857	1.764	1.659

(*a*)

N	$\sigma = 0.52$	$\sigma = 0.6$	$\sigma = 0.7$	$\sigma = 0.8$	$\sigma = 0.9$
4	3.216 819 529	2.856 826 996	2.479 717 191	2.164 280 171	1.895 948 841
5	3.409 571 458	3.000 663 343	2.577 384 837	2.226 683 600	1.930 148 206
6	3.541 533 187	3.097 471 481	2.641 704 488	2.266 464 696	1.950 222 666
7	3.637 181 305	3.166 799 448	2.687 095 748	2.293 897 412	1.963 114 627
8	3.709 447 874	3.218 714 699	2.720 736 861	2.313 882 903	1.971 929 146
9	3.765 810 234	3.258 929 058	2.746 600 781	2.329 047 322	1.978 240 957
10	3.810 888 185	3.290 920 256	2.767 061 526	2.340 920 449	1.982 924 145
T_{ce}	4.169 8	3.547	2.929	2.431	2.004
x_T	1.24	1.21	1.20	1.23	1.90
$\bar{T}_{ce}$	—	1.552	1.426	1.292	1.145
T_c^{NB}	—	1.545	1.420	1.288	1.145

(*b*)

N	$\sigma = 0.92$	$\sigma = 0.94$	$\sigma = 0.96$	$\sigma = 0.98$	$\sigma = 1.0$
4	1.846 973 279	1.799 395 703	1.753 154 153	1.708 190 617	1.664 450 777
5	1.876 150 263	1.823 720 376	1.772 783 992	1.723 271 779	1.675 119 293
6	1.892 680 509	1.836 810 315	1.782 527 743	1.729 754 768	1.678 419 321
7	1.902 924 585	1.844 469 840	1.787 658 012	1.732 403 987	1.678 629 555
8	1.909 677 045	1.849 197 625	1.790 391 588	1.733 167 710	1.677 442 550
9	1.914 331 704	1.852 215 347	1.791 786 399	1.732 948 162	1.675 612 446
10	1.917 649 966	1.854 177 221	1.792 394 734	1.732 200 716	1.675 302 570
T_{ce}	1.928	1.858	1.793 5	1.733	1.678
x_T	2.58	3.75	4.65	—	—

(*c*)

Table 3. Example of VBS extrapolation for (*a*) T_{cN}^{-1} and (*b*) $\nu_N(T_{cN})^{-1}$ at $\sigma = 0.8$.

N	$[N, 0]$	$[N, 1]$	$[N, 2]$	$[N, 3]$
4	0.462 047 388			
5	0.449 098 381	0.428 951 615		
6	0.441 215 785	0.425 256 092	0.411 489 036	
7	0.435 939 286	0.422 792 387	0.411 427 297	0.411 282 231
8	0.432 173 987	0.421 037 831	0.411 383 989	
9	0.429 360 104	0.419 727 779		
10	0.427 182 393			

(*a*)

N	$[N, 0]$	$[N, 1]$	$[N, 2]$	$[N, 3]$
4	0.473 885 812			
5	0.469 184 435	0.459 413 997		
6	0.466 010 369	0.458 171 707	0.454 319 488	
7	0.463 751 125	0.457 332 871	0.454 818 067	0.454 957 397
8	0.462 080 090	0.456 759 995	0.454 926 965	
9	0.460 808 470	0.456 356 621		
10	0.459 819 374			

(*b*)

3.4. *Critical exponent* ν

Data for the critical exponent ν are obtained with less precision due to the numerical derivative involved in their evaluation. The accuracy does not exceed 10^{-6} and VBS extrapolations should be taken with caution.

We present two sets of results for the critical exponent ν. Data ν_N of the first set have been obtained by expanding around the true fixed point T_{cN} corresponding to t^* in equation (8). In the second set of ν_N, the expansion has been made around the extrapolated temperature T_{ce}. In both sets the two extrapolation procedures, LSA and VBS, have been performed for data ν_N^{-1}. The convergence exponent x_ν has been evaluated for each approximation method in particular, inserting the corresponding extrapolation for ν in equation (19). In the presentation of results (tables 4 and 5) only the decimal places which are larger than estimated error bars are retained. Data which could not be evaluated with sufficient precision are omitted.

The MF region where the exact results $\nu = 1/\sigma$ are available is presented in tables 4(*a*) and 5(*a*). For small σ the correction term in equation (24), which is proportional to t_N, becomes important, due to the slow convergence of T_{cN} (see table 2(*a*)). This is compensated by performing the calculations at T_{ce} or by replacing $1/N$ by $(N-1)/N$ in the LSA fit (19) as in our previous article. Except at the edge $\sigma = 0.5$, good agreement within the relative error of less than 9% with exact results is achieved. Poor convergence observed in both cases for $\sigma = 0.5$ is explained by the fact that the present method does not include logarithmic corrections which are important at the point of exchange of MF and the non-trivial fixed points. Concerning the convergence with regard to N, the corresponding exponent x_ν is rather low in the whole MF region.

The non-trivial region is presented in tables 4(*b*), 4(*c*) and 5(*b*), 5(*c*). Like in the MF region, the difference in results is more pronounced between the T_{cN} and T_{ce} sets,

Table 4. Data for ν_N calculated in T_{cN} as a function of σ and N, followed by LSA and VBS extrapolations for ν and for the corresponding convergence exponent x_ν. $\nu_1^{\rm LSA}$ and $x_\nu^{\rm LSA}$ refer to the fit to $(1/N)^x$ form, while $\nu_2^{\rm LSA}$ results from the fit to the form $((N-1)/N)^x$ (see in text). For comparison we cite the exact results ($\nu^{\rm ex}$), the Nagle and Bonner (1970) results ($\nu^{\rm NB}$) and Kosterlitz's (1976) results for ν and y_3 ($\nu^{\rm K}$ and $y_3^{\rm K}$ respectively).

N	$\sigma=0.1$	$\sigma=0.2$	$\sigma=0.3$	$\sigma=0.4$	$\sigma=0.5$
4	2.769 599	2.499 471	2.311 649	2.185 341	2.107 728
5	3.021 370	2.653 906	2.405 228	2.240 506	2.139 220
6	3.257 788	2.792 143	2.485 683	2.286 113	2.163 931
7	3.478 943	2.915 822	2.555 107	2.324 165	2.183 680
8	3.685 798	3.026 827	2.615 422	2.356 260	2.199 726
9	3.879 518	3.126 885	2.668 215	2.383 616	2.212 948
10	4.061 253	3.217 478	2.714 755	2.407 158	2.223 978
$\nu_1^{\rm LSA}$	—	—	4.2	2.9	2.37
$\nu_2^{\rm LSA}$	9.12	4.90	3.41	2.71	2.34
$\nu^{\rm VBS}$	—	7	3.8	2.7	2.32
$\nu^{\rm ex}$	10	5	3.3	2.5	2
$\nu^{\rm NB}$	—	5.0	3.37	2.65	2.22
$x_\nu^{\rm LSA}$	—	—	0.46	0.54	0.72
$x_\nu^{\rm VBS}$	—	0.5	0.57	0.80	1.1

(*a*)

N	$\sigma=0.52$	$\sigma=0.6$	$\sigma=0.7$	$\sigma=0.8$	$\sigma=0.9$
4	2.097 317	2.071 340	2.072 470	2.110 213	2.185 851
5	2.125 449	2.089 963	2.087 531	2.131 358	2.224 638
6	2.147 211	2.103 306	2.097 361	2.145 875	2.254 241
7	2.164 399	2.113 173	2.104 019	2.156 329	2.277 836
8	2.178 221	2.120 643	2.108 634	2.164 127	2.297 242
9	2.189 504	2.126 397	2.111 873	2.170 099	2.313 585
10	2.198 832	2.130 886	2.114 151	2.174 767	2.327 606
$\nu_1^{\rm LSA}$	2.31	2.16	2.123	2.208	2.63
$\nu^{\rm VBS}$	2.26	2.15	2.119	2.198	2.63
$\nu^{\rm NB}$	—	1.98	1.84	1.87	1.98
$x_\nu^{\rm LSA}$	0.79	1.3	2.2	1.2	0.49
$x_\nu^{\rm VBS}$	1.3	2.0	3.4	1.7	0.48

(*b*)

N	$\sigma=0.92$	$\sigma=0.94$	$\sigma=0.96$	$\sigma=0.98$	$\sigma=1.0$
4	2.205 763	2.227 337	2.250 604	2.275 598	2.302 354
5	2.249 737	2.277 121	2.306 858	2.339 015	2.373 663
6	2.283 929	2.316 516	2.352 110	2.390 825	2.432 779
7	2.311 633	2.348 923	2.389 867	2.434 630	2.483 384
8	2.334 753	2.376 334	2.422 199	2.472 574	2.527 692
9	2.354 482	2.400 003	2.450 425	2.506 037	2.567 142
10	2.371 161	2.420 775	2.475 439	2.535 963	2.602 721
$\nu_1^{\rm LSA}$	2.84	3.2	3.8	4.96	8.4
$\nu^{\rm VBS}$	2.87	3.3	3.9	5.05	8.6
$\nu^{\rm K}$	2.5	2.89	3.54	5	∞
$\nu^{\rm NB}$	—	—	—	—	2.20
$x_\nu^{\rm LSA}$	0.41	0.34	0.28	0.23	0.19
$x_\nu^{\rm VBS}$	0.38	0.31	0.26	0.22	0.19
$y_3^{\rm K}$	0.4	0.346	0.283	0.2	0

(*c*)

Table 5. Data for ν_N calculated in T_{ce} as a function of σ and N, followed by LSA and VBS extrapolations. The notations used are the same as defined in table 4.

N	$\sigma=0.1$	$\sigma=0.2$	$\sigma=0.3$	$\sigma=0.4$	$\sigma=0.5$
4	—	25.326 116	9.740 774	6.393 483	4.911 925
5	—	14.634 285	7.478 902	5.301 015	4.245 221
6	55.801 146	11.361 485	6.441 426	4.732 547	3.872 589
7	32.193 529	9.769 804	5.842 530	4.381 115	3.632 084
8	24.386 655	8.826 697	5.451 093	4.140 967	3.462 765
9	20.490 863	8.201 929	5.174 421	3.965 709	3.336 402
10	18.154 078	7.757 068	4.968 017	3.831 711	3.238 061
ν_1^{LSA}	8.64	4.99	3.41	2.672	2.283
ν^{VBS}	8.7	5.06	3.55	2.61	2.19
ν^{ex}	10	5	3.3	2.5	2
x_ν^{LSA}	0.96	0.89	0.79	0.71	0.65
x_ν^{VBS}	0.95	0.92	0.90	0.67	0.57

(*a*)

N	$\sigma=0.52$	$\sigma=0.6$	$\sigma=0.7$	$\sigma=0.8$	$\sigma=0.9$
4	4.703 184	4.080 946	3.486 780	3.017 048	2.575 332
5	4.092 210	3.638 834	3.198 742	2.848 652	2.506 909
6	3.746 776	3.379 462	3.023 878	2.746 166	2.471 810
7	3.522 191	3.206 814	2.904 753	2.676 407	2.452 231
8	3.363 250	3.082 570	2.817 515	2.625 399	2.440 885
9	3.244 152	2.988 286	2.750 378	2.586 201	2.434 295
10	3.151 161	2.913 931	2.696 807	2.554 958	2.430 619
ν_1^{LSA}	2.228	2.134	2.102	2.242	2.424
ν^{VBS}	2.06	2.06	2.05	2.16	2.425
x_ν^{LSA}	0.63	0.63	0.63	0.79	4.18
x_ν^{VBS}	0.50	0.55	0.57	0.61	4.64

(*b*)

N	$\sigma=0.92$	$\sigma=0.94$	$\sigma=0.96$	$\sigma=0.98$	$\sigma=1.0$
4	2.500 271	2.442 374	2.402 853	2.371 697	2.358 362
5	2.450 277	2.411 097	2.391 074	2.380 053	2.388 500
6	2.428 781	2.404 080	2.399 861	2.405 753	2.433 303
7	2.420 121	2.407 482	2.416 904	2.437 811	2.482 883
8	2.417 972	2.415 803	2.437 412	2.472 046	2.533 564
9	2.419 369	2.426 529	2.459 255	2.506 664	2.583 827
10	2.422 780	2.438 399	2.481 412	2.540 847	2.633 029

(*c*)

than between the two extrapolation methods. For $\sigma > 0.9$ (examined in more detail in tables 4(*c*) and 5(*c*)) the results for T_{ce} are non-monotonic and both extrapolation procedures turn out to be ineffective. In contrast to the MF case, there is no simple argument which would favour one of the two sets, and one can only speculate in favour of the set with the larger convergence exponent x_ν, but the above discrepancy extends the error bars to the order of 10%. For comparison we cite (tables 4(*a*), 4(*b*) and 4(*c*)) the results for ν following from the values for the critical exponent γ obtained by finite chain extrapolations (Nagle and Bonner 1970).

The convergence exponent x_ν defined by (19) does not depend very significantly on the method of extrapolation, but rather its value and even dependence on σ is different for T_{cN} and T_{ce} sets of results. It shows faster convergence of T_{ce} data in the MF region, and T_{cN} data in the non-trivial region. It is interesting to point out that close to $\sigma=1$, values of x_ν follow (for $0.92 \leqslant \sigma \leqslant 0.96$) with less than 3% accuracy the predicted expression (26). On the other edge of the non-trivial region, $\sigma \geqslant 0.5$, x_ν shows a tendency to decrease but does not match y_3, which in this limit should be given by (Fisher *et al* 1972)

$$y_3 = 1 - 2\sigma. \tag{27}$$

The value $\sigma=1$ is identified again as a special point. The known essential singularity could not be reproduced using the present method, but a large exponent ν is obtained instead. The results were also not improved by applying the modified procedure of Roomany and Wylde (1980). This is again attributed to the fact that $\sigma=1$ is a rather

Table 6. Data for $d_{\phi N}$ calculated at T_{ce} presented as a function of N and σ, followed by the corresponding VBS extrapolations ($d_{\phi e}$), VBS estrapolations for data taken at T_{cN} ($d_{\phi c}$), exact values and convergence exponent (x_ϕ).

N	$\sigma=0.1$	$\sigma=0.2$	$\sigma=0.3$	$\sigma=0.4$	$\sigma=0.5$
4	0.218 200 5918	0.236 142 6318	0.226 324 4824	0.206 501 3176	0.181 704 2842
5	0.269 708 3717	0.271 767 8067	0.252 905 8466	0.226 433 1681	0.196 229 2562
6	0.302 432 2886	0.294 586 8756	0.270 007 4816	0.239 279 2559	0.205 577 5519
7	0.325 079 2327	0.310 475 3352	0.281 958 3213	0.248 274 8071	0.212 127 3899
8	0.341 687 9551	0.322 182 3523	0.290 790 3660	0.254 936 0795	0.216 984 3617
9	0.354 391 2184	0.331 169 9400	0.297 587 6469	0.260 071 9507	0.220 735 5993
10	0.364 422 4340	0.338 288 5292	0.302 982 5902	0.264 154 6823	0.223 723 0801
$d_{\phi e}$	0.450	0.400	0.350	0.298	0.250
$d_{\phi c}$	0.455	0.401	0.352	0.300	0.242
exact	0.45	0.4	0.35	0.3	0.25
x_ϕ	0.87	0.90	0.90	0.88	0.84

(*a*)

N	$\sigma=0.6$	$\sigma=0.7$	$\sigma=0.8$	$\sigma=0.9$	$\sigma=0.98$	$\sigma=1.0$
4	0.153 945 7188	0.124 377 9968	0.093 704 1333	0.062 441 8211	0.041 730 7016	0.037 395 4443
5	0.163 743 2387	0.129 948 9553	0.095 516 1720	0.061 135 7208	0.039 291 9585	0.034 860 6375
6	0.169 990 3475	0.133 385 9154	0.096 430 1601	0.059 980 4314	0.037 535 6831	0.033 089 8764
7	0.174 348 8358	0.135 736 6042	0.096 961 0198	0.059 028 0219	0.036 248 1094	0.031 821 3718
8	0.177 575 8119	0.137 456 2702	0.097 301 9029	0.058 248 5451	0.035 281 3157	0.030 888 2099
9	0.180 067 9293	0.138 774 9749	0.097 537 8883	0.057 605 6182	0.034 539 5981	0.030 186 2955
10	0.182 054 1056	0.139 821 8729	0.097 711 0131	0.057 069 1906	0.033 960 2991	0.029 648 9965
$d_{\phi e}$	0.200	0.148	0.098	0.055	0.031	0.028
$d_{\phi c}$	0.200	0.142	0.100	0.054	0.029	0.025
exact	0.2	0.15	0.1	0.05	0.01	0.0
x_ϕ	0.78	0.70	0.63	—	—	—

(*b*)

sensitive point since the basic parameter upon which scaling is made becomes marginally relevant due to the exchange of relevance between long-range and short-range interactions (Sak 1973). The changing of regime manifests itself through several other features, such as, for instance, the non-monotonic sequence of T_{cN}, already mentioned in this paper (end of § 3.3).

3.5. Order parameter

Data for the anomalous dimension of the order parameter $d_{\phi N}$, defined by equation (10), have been evaluated at both characteristic temperatures T_{cN} and T_{ce}. Values of the $d_{\phi N}$ have been determined with the same precision as the critical temperature data. Both extrapolation methods have been applied. VBS extrapolations agree with the exact results to one decimal place better than LSA, so we present here only the former. In tables 6(*a*), 6(*b*), and 6(*c*) we present, for different σ, the data d_ϕ taken at T_{ce}, the corresponding VBS extrapolations and VBS extrapolations for the set of data taken at T_{cN}. For comparison, the conjectured exact value for d_ϕ corresponding to $\eta = 2-\sigma$ is given. As can be seen, in the region $0<\sigma<0.9$, d_ϕ has been obtained with very good accuracy, with insignificant difference between results at T_{cN} and T_{ce}. Close to $\sigma = 1$, however, the convergence deteriorates. One can also observe that at $\sigma = 0.9$ the data become non-monotonic in N and for larger σ change into decreasing order. This suggests that the cause lies again in an exchange of relevances occurring at $\sigma = 1$ as already mentioned within the discussion of the exponent ν. As far as the convergence exponent x_ϕ is concerned, the values presented have been obtained by inserting the exact value for d_ϕ into equation (19) and taking the data at T_{cN}. x_ϕ has been evaluated only for $0<\sigma<0.9$ since for $\sigma>0.9$ the data do not fit the expression (19). In spite of very precise results for d_ϕ, it shows slow convergence. Agreement with prediction (25) has not been obtained.

4. Conclusion

In the present paper a more careful analysis of a recently defined finite-range scaling procedure has been performed, taking into account larger ranges and using additional convergence procedures.

A few facts can first be pointed out about the method itself in comparison with FSS, by analogy with which it was constructed.

One is the behaviour in the MF region. While FSS breaks down in MF, this is not true for the FRS. The reason lies in the relevance of the basic scaling parameter. In FSS the size ceases to be a good scaling parameter in the MF region, while within the FRS the range still remains a good one†.

Another one is the behaviour at the edge where the exchange of relevance between the long-range and the short-range interaction fixed points occurs. FRS becomes inapplicable beyond this point, and close to it difficulties are to be expected. For the one-dimensional Ising model considered this point corresponds to $\sigma = 1$.

† Another example of this kind is a different generalisation of the FSS constructed by Botet *et al* (1982) for infinitely correlated systems ($\sigma = -1$). The basic scaling parameter there is the number of particles, and the critical behaviour, which is of MF type by definition, is successfully described.

Concerning the evaluation of critical exponents and T_c for the Ising model considered, rather good agreement with known exact or approximate results is achieved. Including further ranges up to 10 and using the VBS approximation method has improved the precision for T_c and d_ϕ. Results for the convergence exponent show that the convergence is generally rather slow although a fair precision is already reached for small ranges. As in the majority of FSS cases, the convergence exponents do not fit to the values derived from first corrections to scaling. However, it is interesting to point out that in spite of general difficulties near $\sigma = 1$, the convergence exponent x_ν manifests remarkably good agreement with this expected behaviour.

In summary we can conclude that the analysis performed on the one-dimensional Ising model shows that the method is reliable, which justifies the interest in applying it to other problems involving long-range interactions.

Acknowledgments

One of the authors (KU) acknowledges R Jullien for interesting and stimulating discussions. This work has been partially supported by YU-US collaboration program PN 738.

References

Barber M N 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic)
Botet R, Jullien R and Pfeuty P 1982 *Phys. Rev. Lett.* **49** 478
Brézin E 1982 *J. Physique* **43** 15
Brézin E, Zinn-Justin J and Le Guillou J C 1976 *J. Phys. A: Math. Gen.* **9** L119
Bulgadaev S A 1984 *Phys. Lett.* **102A** 260
Dyson F J 1969 *Commun. Math. Phys.* **12** 91
Fisher M E and Barber M N 1972 *Phys. Rev. Lett.* **28** 1516
Fisher M E, Ma S K and Nickel B G 1972 *Phys. Rev. Lett.* **29** 917
Fröhlich J and Spencer T 1982 *Commun. Math. Phys.* **84** 87
Hamer C J and Barber M N 1981 *J. Phys. A: Math. Gen.* **14** 2009
Kosterlitz J M 1976 *Phys. Rev. Lett.* **37** 1577
Nagle J F and Bonner J C 1970 *J. Phys. C: Solid State Phys.* **3** 352
Nightingale M P 1976 *Physica* **83A** 561
Privman V and Fisher M E 1983 *J. Phys. A: Math. Gen.* **16** L295
Roomany H H and Wylde H W 1980 *Phys. Rev.* D **21** 3341
Sak J 1973 *Phys. Rev.* B **8** 281
Simon B and Sokal A 1981 *J. Stat. Phys.* **25** 679
Uzelac K and Glumac Z 1988 *J. Phys. A: Math. Gen.* **21** L421
Uzelac K and Jullien R 1981 *J. Phys. A: Math. Gen.* **14** L151
Vanden Broeck J M and Schwartz L W 1979 *SIAM J. Math. Anal.* **10** 658